1 Vector Spaces

Vectors naturally appear in the discussion of simultaneous linear equations and linear transformations. In this chapter we shall work with the mathematical definition of vector spaces, which is more general than the physical vector spaces that school students are usually familiar with.

1.1 Definition: Vector space

Following Hefferon (see the references in chapter 1), we define A vector space as follows,

A vector space consists of a collection of vectors V with elements $\vec{v_1}, \vec{v_2}, ...$ and a set of scalars S with elements $c_1, c_2, c_3, ...$, and the following four operations,
(1) Addition defined between vectors $(+)$,
(2,3) Addition and multiplication defined between scalars $(\oplus, *)$, and
(4) Multiplication of a scalar with a vector $(\cdot)$.

The two sets V and S are required to satisfy the following conditions. For vector addition, for any $\vec{v_1}, \vec{v_2}, \vec{v_3} \in V$ the following properties are satisfied
(1) Closure under addition, $\vec{v_1} + \vec{v_2} \in V$.
(2) Commutativity $\vec{v_1} + \vec{v_2} = \vec{v_2} + \vec{v_1}$.
(3) Associativity, $(\vec{v_1} + \vec{v_2}) + \vec{v_3} = \vec{v_1} + (\vec{v_2} + \vec{v_3})$.
(4) Existence of a zero vector, $\vec{0}$, such that $\vec{v_1} + \vec{0} = \vec{v_1}$
(5) Existence of additive inverse, for any given $\vec{v} \in V$, there is a $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$.
For scalar multiplications and addition, for any $c1, c2, c3 \in S$, the following properties are satisfied.
(6) Closure under scalar multiplication, $c1 \cdot \vec{v_1} \in V$
(7) Distributivity of addition, $(c_1 \oplus c_2)\vec{v_1} = c_1 \cdot \vec{v_1} + c_2 \cdot \vec{v_1}$.
(8) Distributivity of multiplication, $c_1 \cdot (\vec{v_1} + \vec{v_2}) = c_1 \cdot \vec{v_1} + c_1 \cdot \vec{v_2}$.
(9) Associativity, $(c_1 * c_2)\vec{v_1} = c_1 \cdot (c_2 \cdot \vec{v_1})$
(10) Scalar identity $1 \cdot \vec{v_1} = \vec{v_1}$.

The above definition of vector space is a very general one. However, in general, unless specified I shall consider multiplication between two scalars and between a scalar and vector to have its commonly known interpretation and will not use a multiplication sign for either. Similarly I shall use the same sign + for scalar and vector additions, even though they are mathematically different operations.

1.2 Examples of vector spaces

I discuss three different examples of vector spaces to highlight the fact that vectors need no always be analogous to one's physical idea of vectors. The lattice example below also shows how defining the set of scalars in a different way is useful.

Matrices as vectors:
Consider the set of 2×2 matrices with elements in R. Show that this set satisfies all the properties of a vector space, Take the set of scalars as R.

Polynomials as vectors:
A polynommial for example a third degree one, like $a_0 + a_1x + a_2x^2 + a_3x^3$ can be treated as a four dimensional vector with basis vectors being $1, x, x^2, x^3$. The set S could be Z or R to which the coefficients a_0, a_1, a_2, a_3 belong.

Lattices in crystallography:
If one selects a set of three linearly independent vectors $\vec{b_1}, \vec{b_2}$ and $\vec{b_3}$ and takes all linear combinations using the set of constants as the set of all integers Z, then one can create vector space denoting positions of lattice points in 3D.

There are also many other non-trivial examples of vector spaces, some with unusual definitions of multiplication and additions, however they are more abstract and we shall not discuss them in this introductory course

1.3 Linear independence

The concept of linear independence is useful in understanding dimensions and geometry of vector spaces.

DEFINITION: Linear Independence A set of nonzero vectors $\{\vec{b_1}, \vec{b_2}, ..., \vec{b_n}\}$ is said to be linearly independent, if the linear combination $c_1\vec{b_1} + c_2\vec{b_2} + ... + c_n\vec{b_n}$ can be made $\vec{0}$ if and only if all constants satisfy $c_1 = c_2 = ... = c_n = 0$.

Consider the set $\{(0,1),(1,0)\}$, if one wants to make $c_1(1,0) + c_2(0,1)$ zero, by selecting the constants c_1 and c_2 appropriately, the only possible choice of c_1 and c_2 is that they are both zero, as shown below.

$$\begin{aligned} & c_1(1,0) + c_2(0,1) = (0,0) \\ \Rightarrow\quad & (c_1, c_2) = (0,0) \\ \Rightarrow\quad & c_1 = c_2 = 0. \end{aligned}$$

Here is an example of vectors that are linearly dependent $\{(2,5)(-4,-10)\}$.

$$\begin{aligned} & c_1(2,5) + c_2(-4,-10) = (0,0) \\ \Rightarrow\quad & (2c_1 - 4c_2, 5c_1 - 10c_2) = (0,0)\text{Solving the simultaneous equations} \\ \Rightarrow\quad & c_1 = 2c_2. \end{aligned}$$

Thus any pair of constants c_1, c_2 such that $c_1 = 2c_2$ will make the linear combination $c_1(2,5) + c_2(-4,-10) = \vec{0}$. Since such nonzero constants exist, the vectors are said to be linearly dependent. We can also rewrite the equation as follows,

$$\begin{aligned} & c_1(2,5) + c_2(-4,-10) = (0,0) \\ \Rightarrow\quad & c_1(2,5) = -c_2(-4,-10) \\ \Rightarrow\quad & 2c_2(2,5) = c_2(4,10) \quad \text{substituting } c_1 = 2c_2. \\ \Rightarrow\quad & 2(2,5) = (4,10) \end{aligned}$$

Thus when two vectors are linearly independent, one vector can be written as a multiple of the other vector.

There is an easy way for proving linear independence of many vectors in high dimensions. To understand that method we need to first understand the following result.

If two vectors $\vec{v_1}$ and $\vec{v_2}$ are linearly independent then the pair of vectors $\vec{v_1}, c_1\vec{v_1} + c_2\vec{v_2}$ are also linearly independent, for c_1 and c_2 nonzero real constants. To prove the result, let us assume that $\vec{v_1}$ and $\vec{v_2}$ are linearly independent but $\vec{v_1}, c_1\vec{v_1} + c_2\vec{v_2}$ are linearly dependent. If that is the situation then the following should hold true,

$$\begin{aligned} & a_1\vec{v_1} + a_2(c_1\vec{v_1} + c_2\vec{v_2}) = 0 \quad \text{for} a_1 \neq 0,\ a_2 \neq 0 \\ \Rightarrow\quad & (a_1 + a_2c_1)\vec{v_1} + a_2c_2\vec{v_2} = 0 \end{aligned}$$

Since by assumption $\vec{v_1}$ and $\vec{v_2}$ are linearly independent, The last equation implies that $a_2c_2 = 0$. Substituting in the first equation gives $a_1 = a2c_1 = 0$, by the same argument, which contradicts our initial assumption that $a_1 \neq 0$. Hence $a_1 = a_2 = 0$.

1.4 Row operations

Consider a set of vectors arranged as rows of a matrix.

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1m} \\ a_{21} & ... & ... & a_{2m} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nm} \end{pmatrix}$$

Doing a row operation on the matrix $R_j = R_j + aR_i$, where the jth row gets replaced by a linear combination of jth and ith rows, preserve the linear independence of the two rows as per the argument of the previous section. Thus each row operation creates a new matrix with the same number of independent rows.

In the matrix above, if a_{11} is non-zero then a_{n1} can be replaced by zero by doing the row operation, $R_n = R_n - R_1 a_{n1}/a_{11}$. Similar row operations can make all entries in the column under a_{11} zero. If a_{11} is zero to start with, one can interchange the rows of the matrix, as that does not change the number of independent rows.

Similar row operations can create a matrix so that all entries below the diagonal (or more generally, below the a_{kk} elements) become zero. Such a matrix is called an upper diagonnal matrix. A matrix where all entries above the diagonal entries are zero, an upper diagonal matrix, can also be created using similar row operations.

Once a matrix is in an upper or lower diagonal form, it is easy to show that the number of nonzero digonal entries is the same as the number of independent rows in the matrix. The proof is left as an exercise for the reader.

1.5 Basis set

(DEFINITION) Span : A set of vectors$B = \{\vec{b_1}, \vec{b_2}, ..., \vec{b_n}\}$ is said to span a vector space V if any vector in V can be written as linear combination of vectors in B.

(DEFINITION) Basis set: Given a vector space, a basis set is a smallest set of nonzero, linearly independent vectors which span the vector space.

Examples: The sets $\{(1,0),(0,1)\}$, $\{(-1,0),(0,1)\}$, $\{(1,2),(1,1)\}$ all span the two dimensional Euclidean plane R^2 and each one is a basis set for R^2. The set of vectors $\{(1,0,2),(1,1,3),(2,1,4)\}$ spans R^2 bus has more vectors than necessary hence it is not a basis set.

The dimension of a vector space is the same as the number of vectors in its basis set.

A useful vector space is any plane passing through $(0,0,0)$ in R^3. Consider such a plane given by $x + y - z = 0$. The elements in this vector space are all the vectors of the form (x, y, z) which satisfy the given equation. Since there are three variables and one condition, the dimension of the space is 2. Let us show that all vectors on the plane $x + y - z = 0$ constitute a vector space. We can take the conditions 2,3 and 7 to 10, given in section 1.1 as given, as we are working with ordinary vector additions and real constants and test for rest of the conditions.

Closure (1 and 6): The closure under scalar multiplication and vector addition can be shown by showing that, if $\vec{v_1} = (x_1, y_1, z_1)$ and $\vec{v_2} = (x_2, y_2, z_2)$ satisfy the plane equation then $c_1\vec{v_1} + c_2\vec{v_2} = (c_1x_1 + c_2x_2, c_1y_1 + c_2y_2, c_1z_1 + c_2z_2)$ also satisfy the same plane equation. This is straightforward to show.
Existence of zero (4): Pluggginng in the zero vector $(0,0,0)$ in the plane equation, one sees that it belongs to the plane.
Vector additive inverse (5): For any point (x_1, y_1, z_1) on the plane $(-x_1, -y_1, -z_1)$ is also on the plane.
Thus the position vectors of points on the given plane constitute a vector space.

Since we already know that the plane is a two dimensional, the basis set is expected to have two vectors in it. Any two vectors that are linearly independent and satisfy the plane equation will make up the basis set.

1.6 Applications

In study of mechanics a large number of quantities of interest, like positions, velocities, forces, accelerations are vectors and satisfy differential equations whose solutions form vector spaces. One such example will be discussed in a later chapter.

Storing and processing of information often makes use of techniques of vector analysis. In speech and image processing data is stored as vectors and multiple matrix operations are applied on it to extract features or to make changes. Linear transformations and projections of vectors also form the backbone of best fit methods used to fit experimental data to known models.

Oscillatory systems arise in all branches of science, there are chemical oscillations (for example the Beluozov-Zabotinky reaction), biological oscillations (The heartbeats) and mechanical oscillations (masses attached with springs). Evolution of such systems can be well understood in terms of their normal modes (Basis vectors) and their linear combinations.

Functions, like Polynnomials and trigonometric functions (like $\sin, \cos$ and exponential), form vector spaces. Often complex (as opposed to simple) functions are required to be decomposed in terms of simpler basis functions. One such decomposition, the Fourier series will be discussed later. Such decompositions play important role in applications and analysis.

2 Exercises

1 A line passing through the origin in R^3 can be defined in two ways, (1) With coordinates as set of vectors which are multiple of given vector (2) as intersection of two planes passing through the origin. Define a line using each of this way and show that the position vectors of the points on the line belong to a vector space.

2 Write an equation of plane passing through the origin, Give two different basis sets for the corresponding vector space and show that the vectors in each of the basis set are linearly independent.

3 Show (or give reasoning) that the vectors $(1, 2, 1), (2, 1, 4), (1, 1, 5)$ are linearly independent.

4 Show that the vectors $(2, 1, 3), (1, 3, 1), (5, 0, 8)$ are linearly dependent. Find the equation of plane passing through zero on which all the three vectors lie.