1 Chapter I: Fundamentals

An important part of linear algebra deals with solutions of linear equations. It is useful to be able to see different geometric representations of a set of linear equations and their solutions.

1.1 The row picture

Consider two line equations,

$$\begin{aligned} 2x + y &= 3 \\ x - y &= 0. \end{aligned}$$

A standard way of thinking of the solution of these two simultaneous equations is in terms of the intersection point of the two lines defined by these equations. One usually uses elimination of variables (i.e. in this case multiply the second equation by 2 and subtract it from the first in this case) to solve for the intersection point.

Let us do the following exercise, write two line equations for the following situations for the intersection point (solution). (1) There is a unique solution (2) No solution (3) Infinite solutions. Take some time for discussions before proceeding to the next paragraph.
(1) There is a unique solution: In this case, one needs to make sure that the two lines have different slopes so that they intersect at a point.
(2) No solution:
Here one needs two parallel lines, i.e. lines with the same slopes but different intercept.
(3) Infinite solutions: For this case the two equations need to represent the same lines, same slopes and same intercepts which can be achieved if one equation is a constant multiple of the other equation.

With these geometric pictures it is easy to write down the required equations. To make sure that the slopes are the same one needs to make sure that the ratio of coefficients of the variables x and y is the same. To ensure that the intercepts are same/different, one needs to see that the ratio of the constant term on the right hand side and the coefficient of the variable is same/different. Thus an example of for each of the situation above would be,
(1) A unique solution,

$$\begin{aligned} 2x + y &= 3 \\ x - y &= 0. \end{aligned}$$

Bacause the slope of the first line is -2 and for the second line it is 1. Hence the two lines are not parallel and hence they would cross at a single point.
(2) No solution,

$$\begin{aligned} 2x + y &= 2 \\ 2x + y &= 3. \end{aligned}$$

The two lines have the same slopes -2 but intercepts are different, hence the equations represent parallel lines, hence there is no intersection point.
(3) Infinite solutions,

$$\begin{aligned} 2x + y &= 2 \\ 4x + 2y &= 4. \end{aligned}$$

Here the two equations represent the same line so each point on the line satisfies both the equation, hence in this case there are infinite solutions.

This manner of representing the geometric object specified by each of the linear equations separately is known as the row picture of the linear equations. The same method can also be used for linear equations of three or more variables. A Linear equation of three variables, for example $x + 2y - z = 5$ represents a plane. In the row picture, discuss the three different scenarios in the previous paragraph in terms of intersection of planes.

1.2 The column picture

The two line equations in the previous section,

$$\begin{aligned} 2x + y &= 3 \\ x - y &= 0. \end{aligned}$$

Can also be represented as the following single vector equation.

$$x \begin{pmatrix} 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Satisfy yourself that adding the two vectors and comparing the elements of the left and right sides lead to the same linear equations. However notice that the geometry represented by the above equation is completely different. Here x and y are unknown scaling constants for the known vectors $(2, 1)$ and $(1, -1)$ such that when the scaled vectors are added the resultant vector is equal to the right hand side. Multiplying vectors with numbers and adding them up, like $c_1\vec{v_1} + c_2\vec{v_2} + ... + c_n\vec{v_n}$ is also known as the linear combination of vectors, where $\vec{v_1}, ..., \vec{v_n}$ are vectors and $c_1, ..., c_n$ are constants. Below are the two geometric pictures for the linear equations and their solution. The column picture, while it is not very easy in terms of thinking about the solution, it plays important role in deriving best fit formulas for data-fitting. To draw the column picture one would first need to solve for intersection point coordinates x and y and then plug the values into the vector equation to decide the vectors to be drawn. Note that the column picture can easily be extended for linear systems with multiple variables.

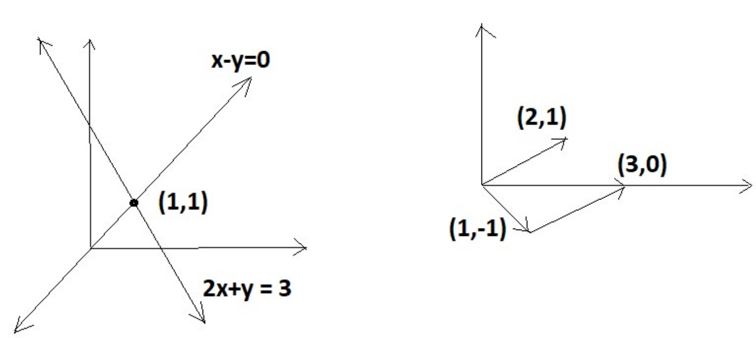

Fig. 1 The row and column pictures for the same linear equations

1.3 The Matrix transformation

One more way of looking at a set of linear equations is in terms of the following matrix equation.

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Again one may carry out the matrix multiplication and compare the vectors on the two sides element wise to show that these are the same linear equations discussed in previous sections. Howevver now one has a new geometric interpretation. Now one is looking for a vector (x, y) which under the linear transformation defined by the matrix gives a new vector $(2, 0)$. Linear transformations are very useful tools in variety of computations ranging from image processing to solving differential equations and will be taken up in greater detail later.

1.4 Dimensions of geometric objects

Consider the equation $x = 3$. What is the dimension of the geometric object defined by the equation? Students generally come up with different answers (1) A point (2) A line (3) A plane. It turns out that all these answers are correct, because the answer to the question depends on the context in which the question is being asked.

If one is working with the real line which is described by a single variable, then $x = 3$ represents a point. In two dimensions, on the x-y plane, the equation represents a straight line parallel to the y-axis. In 3 dimensions (3D), i.e. on R^3, the same equation represents a plane passing through $x = 3$ and parallel to the $y - z$ plane. It is useful to note that the geometric object defined by the equation has one dimension less than the dimensionality of the space. A point is a zero dimensional object, a line is a one dimensional object and a plane is a two dimensional object.

In general the 3D space with variables x, y, z. If no equation is given then all the variables are considered free variables, one can choose any value for each of the variables without violating any condition, because no condition has been specified. Now assume that you are interested in values of x, y, z which satisfies the condition $x + 2y - z = 4$. Now one can not choose any value of x, y, z. However it is still possible to choose any two variable values freely and use the equation to define the third variable. If one chooses $x = 1$ and $y = 4$ then from the equation one can solve to get $z = 5$. Thus one can say that a plane in 3D is defined by two free variables. The dimensionality of the geometric object is the same as number of free variables that can be chosen to define points on it. In general $N_v - N_c = N_d$, where N_v is the number of available space variables, N_c is the number of conditions and N_d is the dimension of the geometric object defined by points satisfying all the conditions (provided such points exist). (**Tangent:** For nonlinear equations a similar idea works. I leave it to students to think about dimensions of a geometric object defined by a nonlinear equation (say $x^2 + y^2 + yz = 2$) and how to get points on the surface.)

Here are examples in two dimensions

(1) All points on $x - y = 0$ make up the one dimensional line. Here $N_v = 2$, $N_c = 1$ and so $N_d = 1$ whereas,

(2) All points satisfying $x - y = 0$ and $x + y = 2$ is a zero dimensional object, the point $(x, y) = (1, 1)$, with $N_v = 2$, $N_c = 2$ and $N_d = 0$ (note that a point is a zero dimensional geometric object).

1.5 Under and over determined systems

All the previous examples of solving simultaneous linear equations were the type where number of variables and number of equations were same. In such cases, as we have seen, either there is a unique solution, or there are infinite solutions or a solution does not exist. What about the case where the number of equations exceed the number of variables? Take for example the following equations,

$$\begin{aligned} 2x + y &= 3 \\ x - y &= 0 \\ 2x - y &= 4 \end{aligned}$$

It should be easy to figure out that for this set of equations there is no point which lies on all the three lines, i.e., a solution to the simultaneous equations does not exist. What about the following ?

$$\begin{aligned} 2x + y &= 3 \\ x - y &= 0 \\ 3x + 2y &= 5. \end{aligned}$$

In this case you should be able to check that there is a unique solutiion. If 3 lines are arbitrarily selected it is unlikely that they would pass through the same point. However the possibility can not be ruled out. In general for a set of 3 or more equations of lines a common intersection point does not exist (even when no pair of lines is parallel). Simultaneous linear equations where the number of equations are more than the number of variables, a solution in general does not exist. Such a set of equations is called overdetermined.

On the other hand, when there are more variables and fewer linear equations, there exist infinite number of solutions. For example for the simultaneous equations,

$$\begin{aligned} 2x + y + z &= 3 \\ x - y - z &= 0 \end{aligned}$$

We know that if the two planes intersect, the intersection set has to be a line. One finds from the second equation that $x = y + z$ and from the first that $x = 1$. Thus the resultant line is defined by the equation, $(x, y, z) = (1, 1 - z, z)$. This equation can also be written as, $(x, y, z) = (1, 1, 0) + z(0, -1, 1)$.

2 Exercises

1 Give an example of equations of three lines where there is a unique solution to the simultaneous equations. How many different geometric scenarios can you consider?
2 Give examples of equations of three lines where there is no solution to the simultaneous equations and (1) Two lines are parallel (2)No two lines are parallel.
3 Select two lines with a unique intersection point and draw the row and column picture.
4 Write the equations selected by you in problems 1, 2 and 3 in matrix form.
5 If for a 2×2 matrix A, the equation $A\vec{x} = \vec{b}$ is to be solved, take examples and discuss possible scenarios where $det(A)$ is zero and $det(A)$ is nonzero.
6 Give examples of equations of three planes with (1) a unique intersection point (2) infinite intersection points (3) No intersection points.
Discuss different ways for getting infinite intersection points and different ways of getting no solution.
7 What is the dimension of the solution of the following simultaneous equations ?

$$\begin{aligned} 2x - y + z &= 3 \\ z &= 2 \end{aligned}$$

What geometrical object does the intersection define?
8 What is the dimension of the solution of the following simultaneous equations ?

$$\begin{aligned} 2x + y + u + v &= 3 \\ y + u &= 1 \end{aligned}$$

What geometrical object does the intersection define?

References

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